Study goals:

- Review on definition (use the idea of limit)
- Differentiation rules (i.e. How to differentiate:)
  - Polynomials ax^n-n can be positive, negative or fractions
  - Products $f(x)g(x)$
  - Fractions $\frac{f(x)}{g(x)}$
- Equations of tangents & normals to the curve
- How to judge a function is increasing or decreasing
- ★ Turning (i.e. Stationary) points + maximum/minimum points
  - Second-order derivative
- Real-world problem application

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Core content	Extension content
	1. understand and use the gradient function $\frac{dy}{dx}$ 2. differentiation of kx^n where n is a positive integer or 0, and the sum of such functions

positive
negative
fraction.

Notes: including expressions which need to be simplified first.

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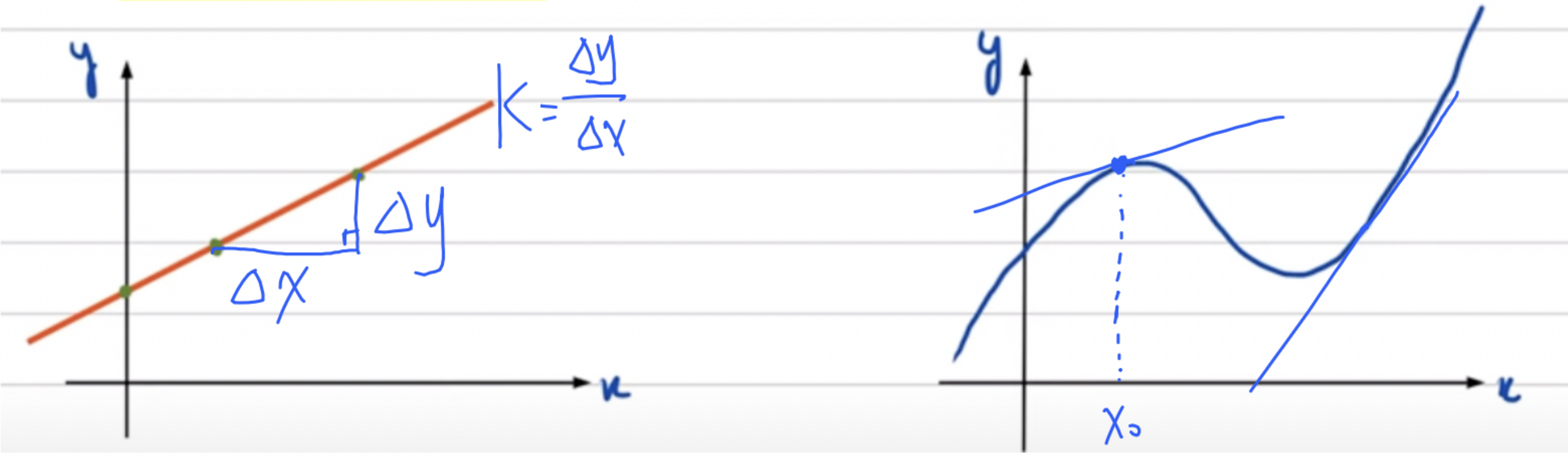
Core content	Extension content
	use of differentiation to find stationary points on a curve: maxima, minima and points of inflection sketch a curve with known stationary points

Lecture 1:

1. Review on definition



- The gradient of a line segment is

$$\frac{\text{Change in the } y \text{ direction}}{\text{Change in the } x \text{ direction}}.$$

Question:

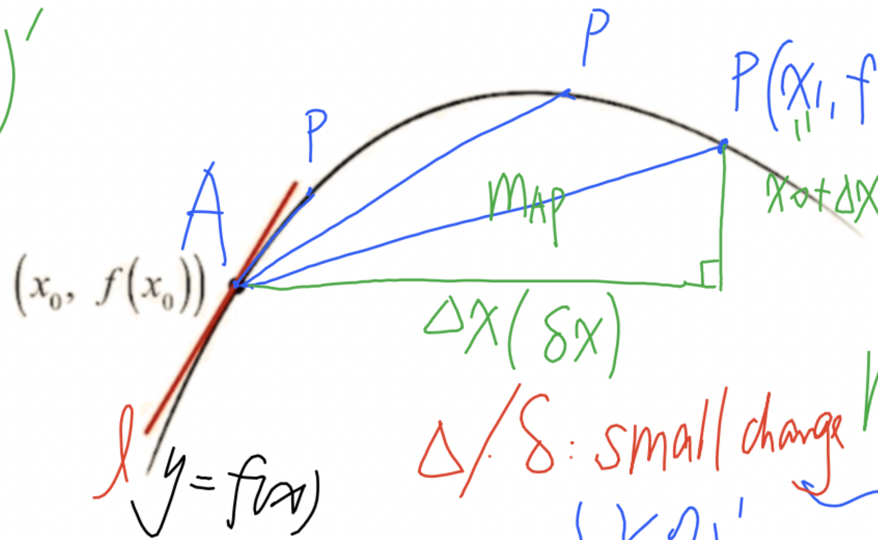
Find the equation of the tangent to the curve $y = x^3$ at the point (1, 1).

$\lim_{\Delta x \to 0}$ ← when $\Delta x \to 0$

Solution:

Solve the tangential problem with the idea of $\boxed{\text{limit}}$ 极限 chord弦

$(Kx^n)'$



$$m_{AP} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$m_l = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

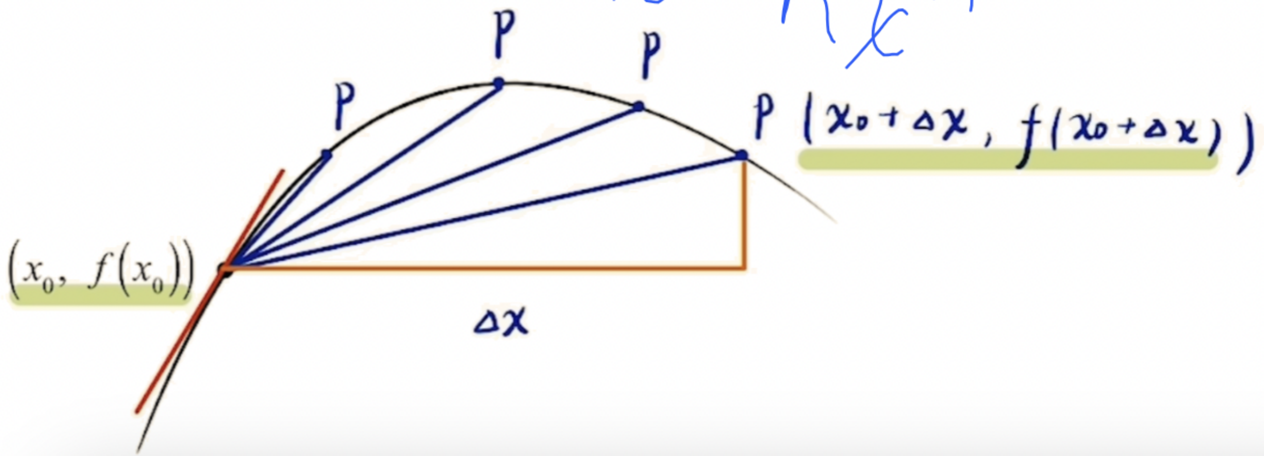
Δx

Δy

Δt

Δ/δ: small change

$(x^n)' = nx^{n-1}$



- Still look at this example:

  Find the equation of the tangent to the curve $y = x^{\cancel{3}\,2}$ at the point ~~(1, 1)~~. $A(x_0, y_0)$

$f(x) = x^2$

$$m_l = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x_0+\Delta x)^2 - x_0^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x_0^2} + 2x_0\Delta x + (\Delta x)^2 - \cancel{x_0^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x_0 + \underbrace{\Delta x}_{\to 0}) = 2x_0 = 2$$

$(x_0, y_0)\,.\,(x_1, y_1)$

$y = mx + b$

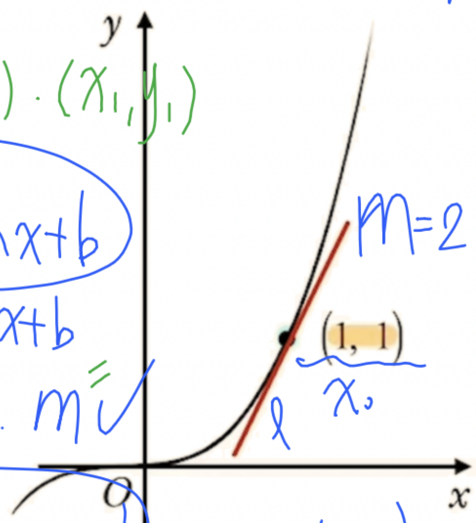
$= 2x + b$

$(x_0, y_0)\,.\ m$

$y - y_0 = m(x - x_0)$

$y - 1 = 2(x - 1)$

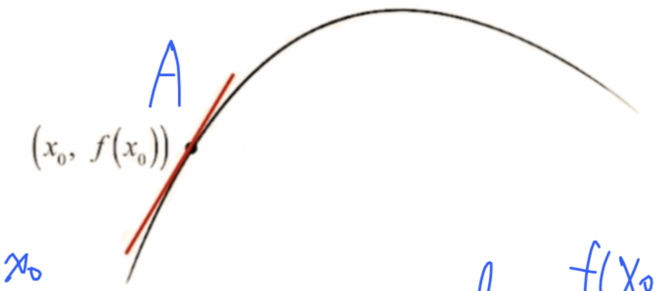
$y = 2x - 1$



Definition:

y'

$y'|_{x=x_0}$



$$\left.\frac{dy}{dx}\right|_{x=x_0} \text{ or } y'|_{x=x_0}$$

Hence, $k = f'(x_0) = y'|_{x=x_0} =$ $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.

And $\frac{dy}{dx}$ is called the gradient function.

The symbol, δ, is used to mean a small change.

Summary:

- The gradient of a graph at a point is an **instantaneous rate of change.**
- This instantaneous rate of change is called a **derivative**.
- The process of finding a derivative is called **differentiation.**
- If y is a function given in terms of x, then the derivative of y is written either as y', or as $\frac{dy}{dx}$. $f'(x)$

Examples:

If $y = x^3$, find the value of y' at $x = 3$ $\quad y'|_{x=3}$

$y'|_{x=x_0}$: value $\quad y' =$ ___ function

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \boxed{\lim_{\Delta x \to 0} \frac{(3+\Delta x)^3 - 3^3}{\Delta x}} = \lim$

$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$

$(a+b)^4 = a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + a^0b^4$

Pascal Triangle

1 $\quad n=0$

1 1 $\quad n=1$

1 2 1 $\quad n=2$

1 3 3 1 $\quad n=3$

1 4 6 4 1

1 5 10 10 5 1

2. **Differentiating polynomials**

Try this table:

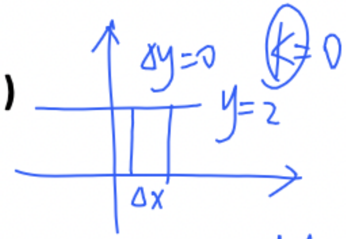
3. Differentiating constants and multiples of x

$a\cdot x^n$

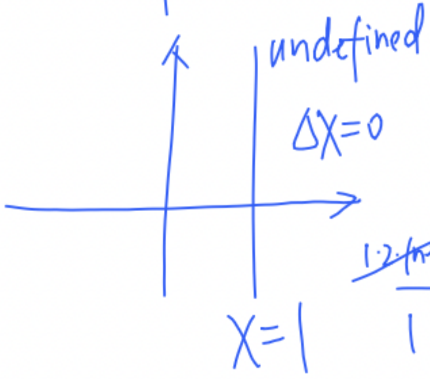
- **Constants**

$y=2$

$y = c$ ===> then $\frac{dy}{dx} =$ ____0____. (Think of the graph)



- **Multiples of x**



Now, if y is a constant multiple of a function of x,
i.e. $y = ax^n$, what is $\frac{dy}{dx}$? $n\cdot a\cdot x^{n-1}$

$f(x)$

We can use definition: $y' = limit_{\delta x\to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$, $\delta x = h$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

n choose r

$$= \lim_{h \to 0} \frac{a(x+h)^n - ax^n}{h}$$

$$= a \lim_{h\to 0} \frac{\cancel{x^n} + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n - \cancel{x^n}}{h}$$

$$= a \lim_{h\to 0} \binom{n}{1} x^{n-1} + \cancel{x^{\cdot\cdot}h} + \cdots + \cancel{h^{2}} + \cdots \cancel{h^{n-1}}$$

$\binom{n}{1} = n$

$= a \cdot n \cdot x^{n-1}$

$\frac{1\cdot 2 \cdots \cancel{(n-1)}(n)!}{1!\,\cancel{(n-1)}!}$

$3! = 1\cdot 2\cdot 3$

Example:

If $y = x^3 + 4x^2$, find the value of y' when $x = 3$

$3x^2$ $\quad$ $8x$

For $4x^2$, a=__4__, n=__2__. Then $\frac{d}{dx}4x^2 =$ ____$8x$____.

$(a\cdot x^n)' = a\cdot n x^{n-1}$

$y' = 3x^2 + 8x$

Lecture 2

4. **Differentiating products and fractions** $\frac{f}{g}$ $(ax^n)' = a \cdot n \cdot x^{n-1}$

AS level: split into separate terms

Example:

1. **Product**

Find $\frac{dy}{dx}$ when $y=(x-3)(x^2+7x-1)$.

$y=c$

$y'=0$

$$y=(x-3)(x^2+7x-1)=\underbrace{x^3+4x^2-22x+3}_{}$$

$3x^2+8x-22$

$$\Rightarrow \quad \frac{dy}{dx}=3x^2+8x-22$$

2. **Fraction**

Find $\frac{dt}{dz}$ when $t=\frac{6z^2+z-4}{2z} = 3z + \frac{1}{2} - \frac{2}{z}$

$(ax^n)' = a \cdot n \cdot x^{n-1}$

$$t=\frac{6z^2+z-4}{2z}=\frac{6z^2}{2z}+\frac{z}{2z}-\frac{4}{2z}$$

$(2 \cdot z^{-1})' = -2z^{-2}$

$$=3z+\frac{1}{2}-\frac{2}{z}$$

$\frac{dt}{dz} = 3 + 2z^{-2}$

x^{-2}

****Extension: product and quotient rule**

$y = \_\_\_$

$$\frac{d}{dx}[f(x)g(x)] \overset{积}{=} f(x)g'(x) \overset{商}{+} f'(x)g(x)$$

$\frac{dy}{dx}$ vs $\frac{d}{dx}(\_)$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$$\boxed{\frac{f' g - f g'}{g^2}}$$

$\frac{x-3}{x^2+7x-1}$ $= \frac{1 \cdot (x^2+7x-1)-(x-3)(2x+7)}{(x^2+7x-1)^2}$

Example:

Find $\frac{dy}{dx}$ when $y=(x-3)(x^2+7x-1)$. $= f'g + f \cdot g' = x^2+7x-1+(x-3)(2x+7)$

Practice:

Differentiate each of the following equations with respect to the variable concerned.

3. $y=(3x-4)(x+5)$
4. $y=(4-z)^2$
5. $s=\dfrac{t^{-1}+3t^2}{2t^2}$
6. $s=\dfrac{t^2+t}{2t}$
7. $y=\left(\dfrac{1}{x}\right)(x^2+1)$
8. $y=\dfrac{z^3-z}{\sqrt{z}}$
9. $y=2x(3x^2-4)$
10. $s=(t+2)(t-2)$
11. $s=\dfrac{t^3-2t^2+7t}{t^2}$
12. $y=\dfrac{\sqrt{x}+7}{x^2}$