Matrices

- A **matrix** is an array of numbers, usually written in brackets.
- The **order** of a matrix is its size, given in terms of the number of rows and the number of columns. Matrices can be any size, but you will only **need** to work with those that are 2×1 or 2×2.

Vectors such as $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ are 2×1 **matrices**.

row column

$a \times b$ 5×4

$$\begin{array}{c} \\ ① \\ ② \end{array}\begin{array}{c} \begin{array}{cc} & \end{array} \\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ \begin{array}{cc} ① & ② \end{array} \end{array} \; a \times b = 2 \times 2$$

Types of matrices

square matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$$

n x n matrix. 2×2

column matrix

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

n x 1 matrix. 2×1

row matrix

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

1 x n matrix 1×2

zero matrix

$$0_{mn} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} 2 \times 2$$

Matrix operation

1. Addition / Substraction

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$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \overset{A_{11}}{39} & \overset{A_{12}}{41} & 33 \\ 38 & 32 & 19 \end{bmatrix} + \begin{bmatrix} \overset{B_{11}}{46} & \overset{B_{12}}{37} & 38 \\ 26 & 18 & 26 \end{bmatrix} = \begin{bmatrix} 85 & 78 & 71 \\ 64 & 50 & 45 \end{bmatrix}$$

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2×3 2019 $\quad$ 2011 2×3 $\quad$ 2×3

Rules

- We can **only** Add or Substract matrices with **same order**.
- Add/Substract **corresponding** elements.

EXAMPLE 1

Which of the following a - e do you think you can add/subtract. Why?
What do you think the answer would be?

a $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$ ✓

b $\underbrace{(3 \quad -1)}_{1\times 2} + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{2\times 1}$ ✗

c $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}$ ✓

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e & g \\ f & H \end{pmatrix} = ($$

2. Multiplication

- Multiplication
  - Scalar Multiplication
  - Matrix Multiplication

2.1 Scalar Multiplication

Scalar = a real number

Rules	Mutilpy each element with a scalar

For example, if $\mathbf{A} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times 2}$, then $k\mathbf{A} = k\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which is $\begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$.

EXAMPLE 2

$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\quad \mathbf{B} = (6 \quad 0 \quad -4)$

$2A = \begin{pmatrix} 2 & 4 \\ -2 & 0 \end{pmatrix}$ $\quad \frac{1}{2}B = (3 \quad 0 \quad -2)$

Find **a** $2\mathbf{A}$ **b** $\frac{1}{2}\mathbf{B}$

2.2 Matrix Multiplication

Condition

$A_{1\times 2} \neq B_{1\times 2}$

AB ✗

EXAMPLE 3 $A = (3 \quad -1)$ $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

1×2 2×1

$\underline{m} \times \underline{n}$

行 列

$$\mathbf{AB} = \mathbf{A} \times \mathbf{B} = 1 \times 1$$

$$1 \times \boxed{2} \quad \boxed{2} \times 1$$

$A_{2\times 3}$ $B_{3\times 4} = 2\times 4$ ✓

1st 列 number of **columns**	=	2nd 行 number of **rows**

$A \cdot B = 1 \neq BA = [\quad]_{2\times 2}$

2×3 4×5 2×1 1×2

order matters

$$\boxed{\mathbf{AB} \neq \mathbf{BA}}$$

Matrices are **NOT** **Commutative** under multiplication

To multiply matrices, multiply each row from the first matrix by each column from the second matrix. When multiplying a row by a column the elements are multiplied in pairs and the results added.

Examples:

$$\begin{pmatrix} 4 & 5 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \times 3 + 5 \times 2 \\ 2 \times 3 + 6 \times 2 \end{pmatrix} = \begin{pmatrix} 12 + 10 \\ 6 + 12 \end{pmatrix} = \begin{pmatrix} 22 \\ 18 \end{pmatrix}$$

$\begin{pmatrix} 22 \\ 18 \end{pmatrix}_{2\times 1}$

$$\begin{pmatrix} 2 & 3 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 2 \times 7 + 3 \times 5 & 2 \times 1 + 3 \times 4 \\ 0 \times 7 + 6 \times 5 & 0 \times 1 + 6 \times 4 \end{pmatrix}$$

2×2 2×2 ✓ 2×2

$$= \begin{pmatrix} 14 + 15 & 2 + 12 \\ 0 + 30 & 0 + 24 \end{pmatrix} = \begin{pmatrix} 29 & 14 \\ 30 & 24 \end{pmatrix}$$

$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 5 \\ 6 \end{pmatrix}} = \begin{pmatrix} 1\times 5 + 2\times 6 \\ 3\times 5 + 4\times 6 \end{pmatrix}_{2\times 1}$

$\underset{1\times 2}{\begin{pmatrix} 1 & 2 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 3 \\ 4 \end{pmatrix}} = \begin{pmatrix} 1\times 3 + 2\times 4 \end{pmatrix}_{1\times 1} = 11$

Step 1. Determine the size of each Matrix

Step 2. Can this multiplication be performed?

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$
$$(n \times m) \times (m \times k) \quad (n \times k)$$

Step 3. Mark up an empty Matrix of the correct size

前列等后行

$$\begin{pmatrix} \_ & \_ \\ \_ & \_ \end{pmatrix}_{n \times k}$$

Step 4. Write in calculations

Step 5. Calculate

$$\begin{pmatrix} 2y \\ 5x-3y \end{pmatrix}_{2\times1} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} x = \_ \\ y = 2 \end{cases}$$

Example:

Use these matrices.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix}_{2\times2} \quad C = \begin{pmatrix} 4 \\ -1 \end{pmatrix}_{2\times1} \quad D = \begin{pmatrix} x \\ y \end{pmatrix}_{2\times1}$$

a Calculate **i** AB **ii** BA **iii** AC

b Show that AI = A and IC = C.

c Given that BD = C find the values of x and y.

2×2 2×1

a. i AB $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 0+15 & 2-9 \\ 0+20 & 4-12 \end{pmatrix}$

ii BA $\begin{pmatrix} 0 & 2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0+4 & 0+8 \\ 5-6 & 15-12 \end{pmatrix}$ $\begin{pmatrix} 15 & -7 \\ 20 & -8 \end{pmatrix}$ ✓

iii AC $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $= \begin{pmatrix} 4 & 8 \\ -1 & 3 \end{pmatrix}$ ✓

$= \begin{pmatrix} 4-3 \\ 8-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ ✓

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad n \times n$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculate

a $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ 7 & 8 \end{pmatrix}$

b $\begin{pmatrix} 3 & 2 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 9 & 7 \end{pmatrix}$

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$$\mathrm{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \overset{M^{-1}}{\mathrm{N}} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

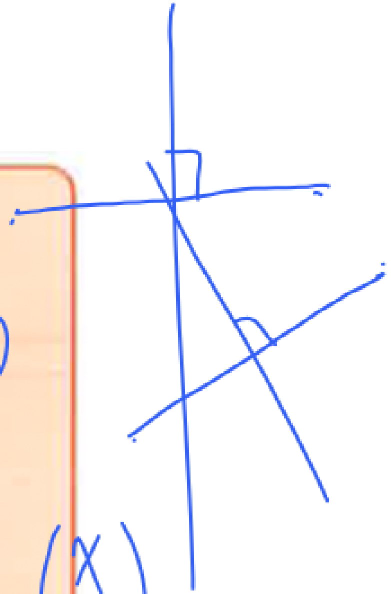
2x2

Show that NM = I and MN = I

Transformation

Review:

To describe a	give
Reflection	The position of the $\boxed{\text{mirror line}}$
Rotation	1 The angle of rotation 2 The direction (clockwise or anti-clockwise) 3 The centre of rotation
Translation	The vector or the distance and direction $\begin{pmatrix} x \\ y \end{pmatrix}$
Enlargement	1 The scale factor $SF>1$ or $SF<1$ 2 The centre of enlargement



How do I transform a point using a matrix?

- A point (x, y) in a 2D plane can be **transformed** on to another point (x', y') by a **matrix, M**
  - (x, y) is the **object** and (x', y') is the **image**
- The **coordinates** of the image point can be found using **matrix multiplication**
- To transform (x, y) by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2x2
  - Write (x, y) as a **column vector,** $\begin{pmatrix} x \\ y \end{pmatrix}$
  - Use **matrix multiplication** to work out $\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{2\times2}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{2\times1}$, which gives $\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}_{2\times1}$
  - **Write down** the image point **coordinates**, (x', y')

$A'(x', y')$

$B(m', n')$

$\begin{pmatrix} x \\ y \end{pmatrix}$

(x, y) A

C

(m, n) B

$\begin{pmatrix} m' \\ n' \end{pmatrix}$ $\begin{pmatrix} m \\ n \end{pmatrix}$

Example:

The transformation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ 2x1 P' maps triangle $P(2, 1)$ $Q(3, 1)$ $R(2, 4)$ to $P'Q'R'$.

a Draw PQR and $P'Q'R'$. **b** Give a full description of the transformation.

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Combining transformation matrices

If **A** and **B** are transformation matrices and P is a point or shape

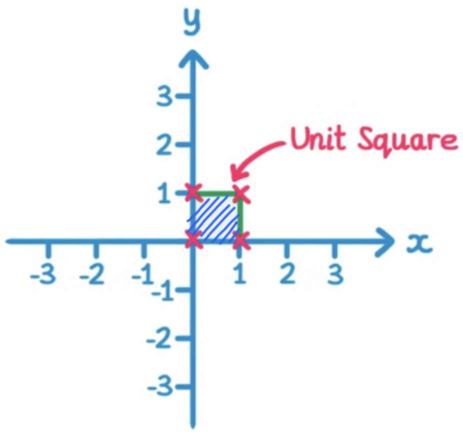
- **A**(P) is the image of P after transformation **A**.
- **BA**(P) is the image of **A**(P) after transformation **B**.
- **BA** is the matrix that represents **A** ***followed by*** **B**.

Order is important when transformation matrices are combined:

AB is **B** *followed by* **A**.

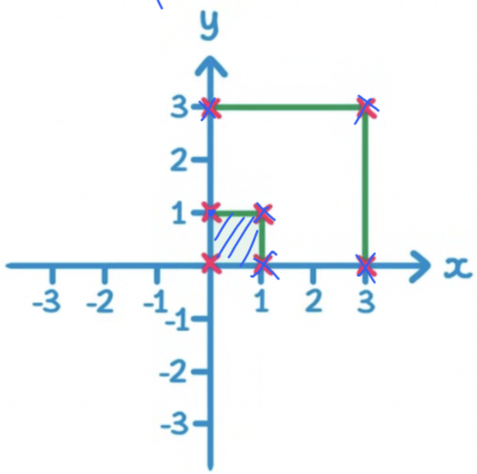
$\boxed{B\boxed{A(P)}}$

$A B(P)$



1. Enlargement

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\Bigg(\qquad = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

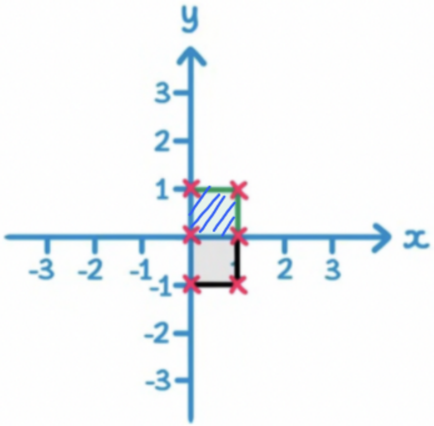


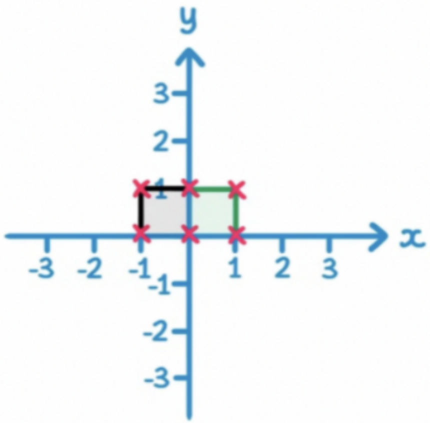
$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Represents an enlargement, scale factor k, about the origin.

2. Reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

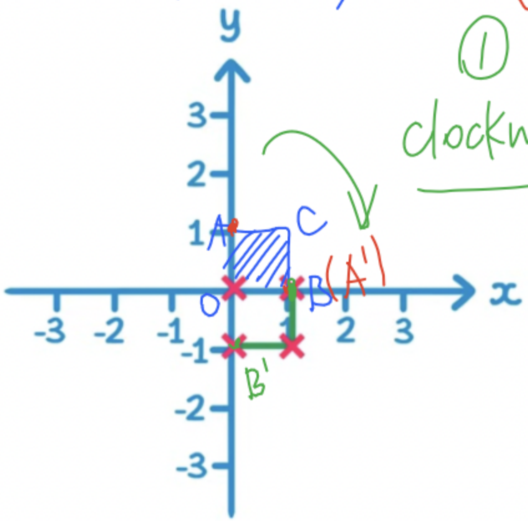




3. Rotation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$



(1) clockwise rotation by 90°, (2)

center of rotation : (0, 0). (3)

$$\begin{array}{cc} B & A \\ \end{array}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

a Find the matrix that represents **i** reflection in the x axis **ii** reflection in $y = -x$
iii reflection in the x axis followed by reflection in $y = -x$.

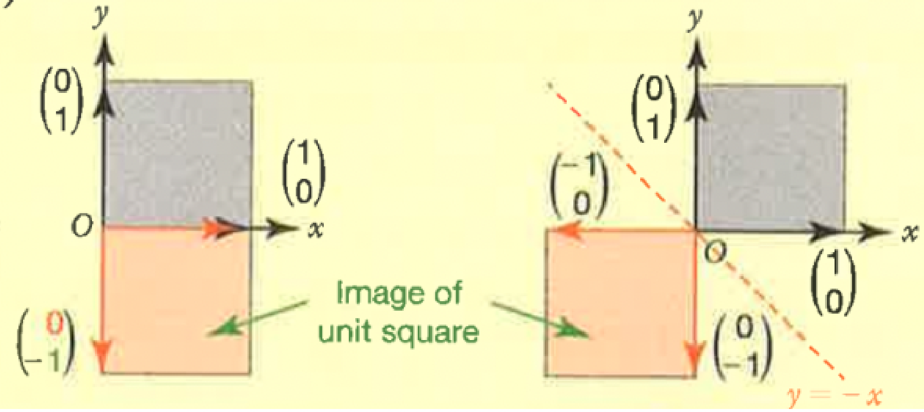
b Describe the single transformation that is equivalent to the combined transformation.

a **i** ① $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Sketch what happens to the unit square.

Matrix = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

ii $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Matrix = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



② **iii** Multiply the matrices, but take care with the order.

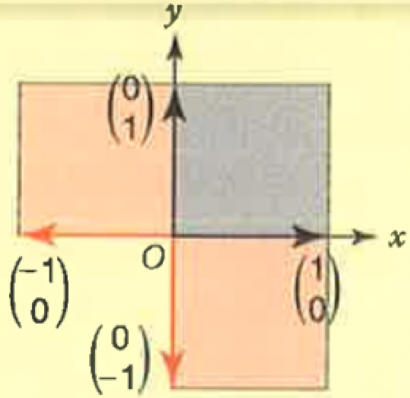
$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

b ① $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

③ This is rotation through 90° clockwise about (0, 0).

Interpret $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in terms of transformations.

(1)(2)(3) Find and describe the transformation represented by each matrix.



$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ rotates the unit square through 90° clockwise about (0, 0).

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotates the unit square 90° anticlockwise about (0, 0).

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix – it maps the unit square onto itself.

A rotation of 90° anticlockwise about (0, 0) followed by a rotation of 90° clockwise about (0, 0) returns an object to its original position.