3.3.3 Transformations, Matrices and Vectors

C18.

G21

Core content	Extension content
describe and transform 2D shapes using single rotations, reflections, translations, or enlargements by a positive scale factor and distinguish properties that are preserved under particular transformations	including combined transformations and enlargements by fractional and negative scale factors

Notes: translations will be specified by a vector.

G22

Core content	Extension content
	understand and use vector notation; calculate, and represent graphically the sum of two vectors, the difference of two vectors and a scalar multiple of a vector; understand and use the commutative and associative properties of vector addition; solve simple geometrical problems in 2D using vector methods

G23

Core content	Extension content
	multiplications of matrices

Notes: multiplying a 2×2 matrix by a 2×2 matrix or by a 2×1 matrix, multiplication by a scalar.

G24

Core content	Extension content
	the identity matrix, **I**

Notes: 2×2 only.

G25

Core content	Extension content
	transformations of the unit square in the $x - y$ plane

Notes: representation by a 2×2 matrix
transformations restricted to rotations of 90°, 180° or 270° about the origin, reflections in a line through the origin (ie $x = 0$, $y = 0$, $y = x$, $y = -x$) and enlargements centred on the origin.

G26

Core content	Extension content
	combination of transformations

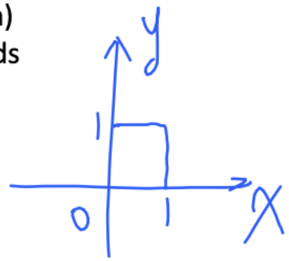
Notes: using matrix multiplications

use of **i** and **j** notation is not required.

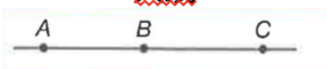
Study Goals:

① enlarge
② rotation
③ reflection
④ translation

- Pythagoras Theorem
- Trigonometric
  - Sine rule
  - Cosine rule
  - Area of any triangle
- Vector size✓. direction ✗
  - Vector calculation (addition, subtraction, scalar multiplication) size✓. ~~direction~~
  - Solve simple geometrical problems in 2D using vector methods
- Matrices
  - Definition and types (e.g.2x2 identity matrix)
  - Matrix multiplication (two types) scalar / matrix
- Transformation
  - Transformations of the unit square in the $x - y$ plane
  - Combination of transformation using matrix multiplication

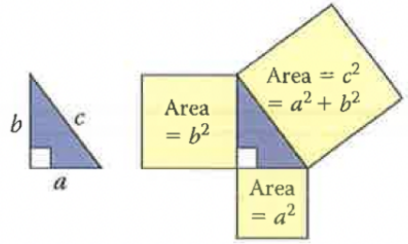


Vocabularies

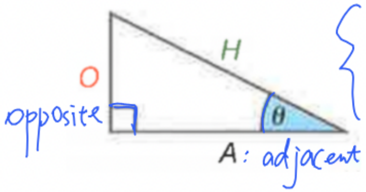
Name	Translate
Hypotenuse	斜边
Pythagoras' theorem	勾股定理
Identity matrix	单位矩阵
scalar	标量 （只有大小，没有方向的量）
vector	向量
colinear	共线 
Slant edge	斜棱

Pythagoras Theorem

- In symbols, Pythagoras' theorem is
  $c^2 = a^2 + b^2$ where c is the length of the hypotenuse.



Trigonometry

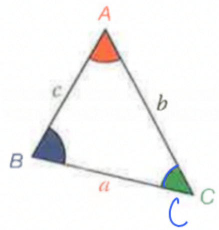


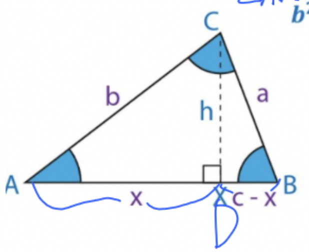
$$\sin\theta = \frac{O}{H}$$
$$\cos\theta = \frac{A}{H}$$
$$\tan\theta = \frac{O}{A}$$

There are two rules that you can use to work out sides and angles in triangles that are *not* right-angled and a formula you can use to find the area of any triangle.

1. **Sine rule**	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	(1) At least one side & (2) opposite ∠
or	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	
2. **Cosine rule**	$a^2 = b^2 + c^2 - 2bc\cos A$	Two sides & ∠ between
or	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$	Three sides
Area of any triangle	$= \frac{1}{2}ab\sin C$	Two sides & ∠ between

$A = \cos^{-1}(\ )$





ΔADC $b^2 = h^2 + x^2$

ΔCDB $a^2 = h^2 + (c - x)^2$

$$b^2 - x^2 = a^2 - (c - x)^2$$

$$a^2 = b^2 + c^2 - 2cx$$

$$CosA = \frac{x}{b}, \therefore x = bCosA$$

$$a^2 = b^2 + c^2 - 2bcCosA$$

$$a^2 = b^2 + c^2 - 2bcCosA$$

Example:

The base *BCDE* of this pyramid is a square with sides of length 8 cm.

The length of each slant edge is 10 cm.

F is the centre of the base and *M* is the mid-point of *CD*.

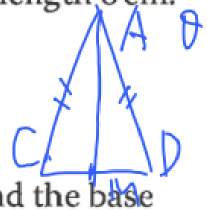
Calculate **a** the height of the pyramid AF

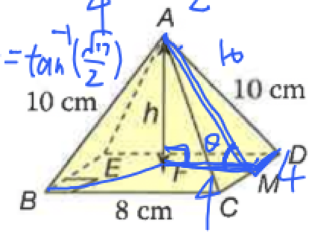
b $\angle ABF$, the angle between a slant edge and the base

c $\angle AMF$, the angle between a triangular face and the base. 64°

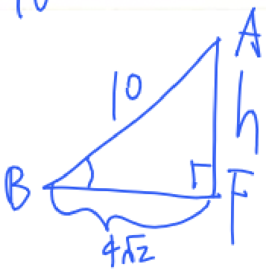
$\tan\theta = \frac{h}{4} = \frac{\sqrt{17}}{2}$

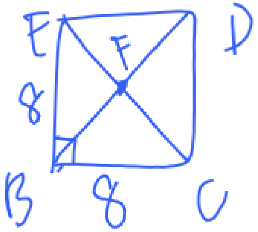
$\theta = \tan^{-1}\left(\frac{\sqrt{17}}{2}\right)$





$\sin = \frac{h}{10}$





$EC = \sqrt{2 \cdot 8^2} = 8\sqrt{2}$ cm

$BF = \frac{1}{2}EC = 4\sqrt{2}$ cm.

$h = \sqrt{100 - 16 \cdot 2}$

$= 2\sqrt{17}$ cm

Vectors

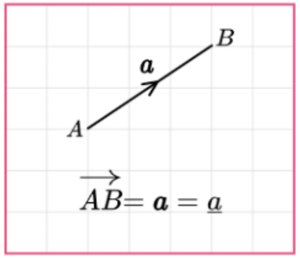
1. Definition

- Vectors describe a movement from one point to another.
- Two characteristic
  - Direction
  - Magnitude: 'how large' something is. length/size
    - How to find: the length of the line segment

Scalar : number

2. Vector notation

This diagram shows a vector representing the move from point A to point B.



$\overrightarrow{AB} = \mathbf{a} = \underline{a}$

- Using an **arrow**
- Using **boldface**
- **Underlined**

Vector $\boldsymbol{a}$ can be written as the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$

<0 >0

>0

<0

> The first number in a column vector is called the *x* **component** and the second is called the *y* **component**.

The first number:

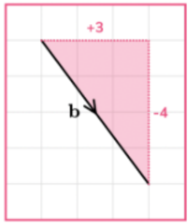
- **positive**, the direction is to the right.
- **negative**, the direction is to the left.

$\underline{a} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ right down

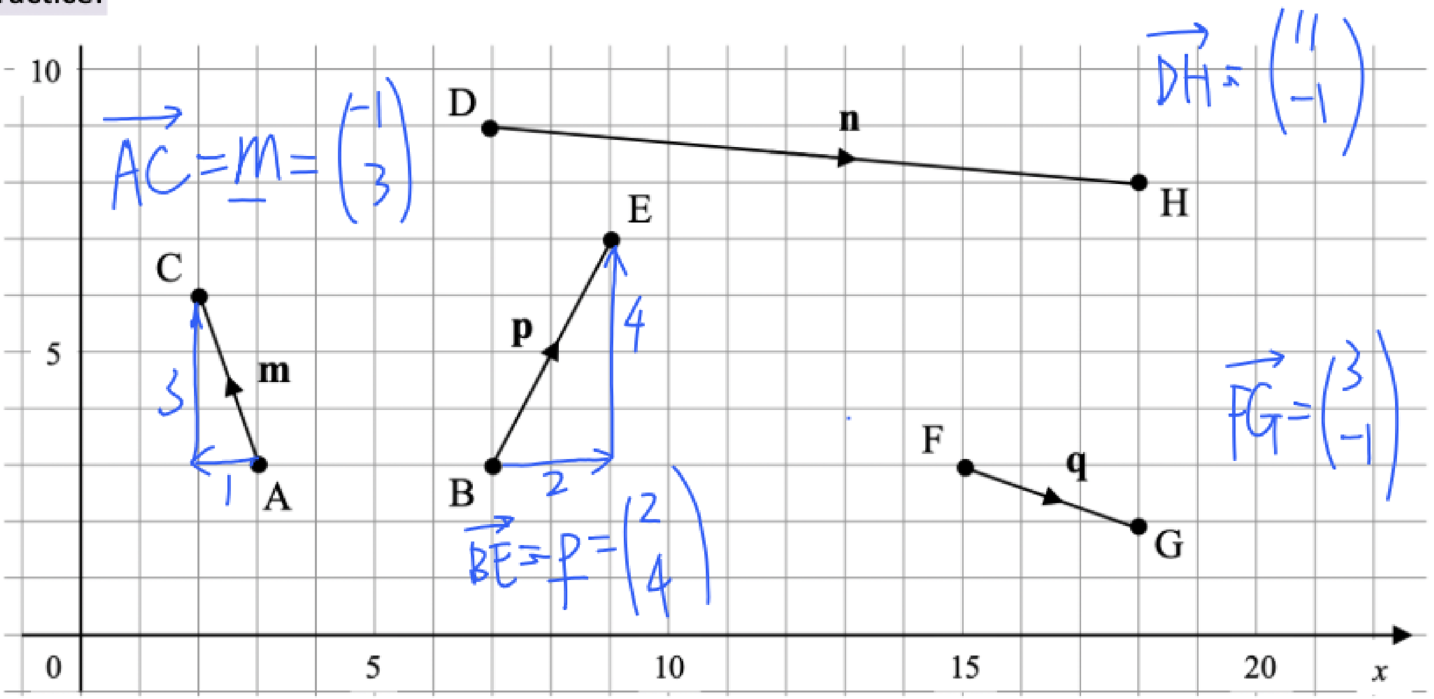
The second number:

- **positive**, the direction is upwards.
- **negative**, the direction is downwards.

Vector $\boldsymbol{b}$ can be written as the column vector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$



Practice:

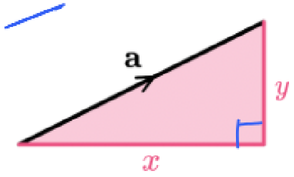


$\overrightarrow{AC} = \underline{m} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$
$\overrightarrow{DH} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}$
$\overrightarrow{BE} = \underline{p} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
$\overrightarrow{FG} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$
A
B
C
D
E
F
G
H
m
n
p
q
1
2
3
4
0
5
10
15
20
x

3. Magnitude of a vector

The length of a vector is its absolute value and we use the modulus symbol.

$a = \begin{pmatrix} x \\ y \end{pmatrix}$

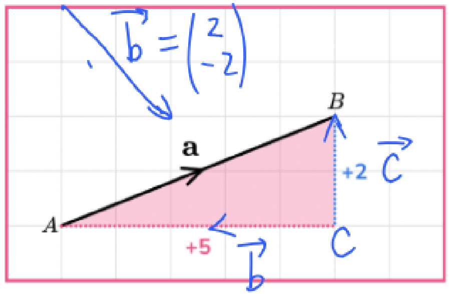


$|a| = \sqrt{x^2 + y^2} = \bigcirc$ 模长

value

$|a| = \sqrt{x^2 + y^2}$

E.g.



$\underline{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\vec{CA} = \vec{b}$

$\vec{AB} = \underline{a}$

$\vec{BA} = -\underline{a}$

$\vec{AB} = \vec{AC} + \vec{CB}$

$= -\vec{b} + \vec{c}$

$|a| = \sqrt{x^2 + y^2} = \sqrt{5^2 + 2^2} = \sqrt{29}$

If the magnitude is equal to 1, then the vector is known as a **unit vector.**

If the magnitude is equal to 0 , then the vector is known as a **zero vector**.

Practice:

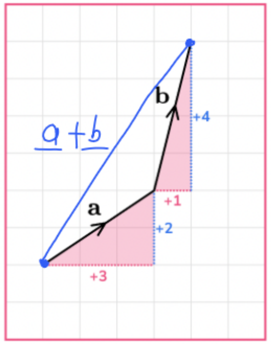
Find the magnitude of vector a, giving your answer to 1 decimal place:

$a = \begin{pmatrix} -4 \\ 5 \end{pmatrix} \Rightarrow |\underline{a}| = \sqrt{(-4)^2 + 5^2} = \sqrt{41} \approx 6.4$ (1 dp).

(6.4)

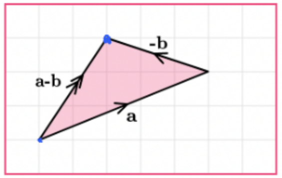
Vector arithmetic 向量运算

4. Vector addition and subtraction



$\underline{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. $\underline{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

$\underline{a} + \underline{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

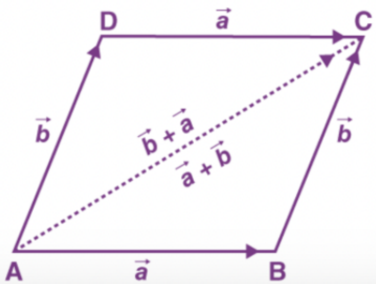


$a - b = \begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Properties of vector addition

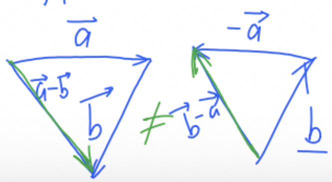
(1) Commutative 交换律

- **a** + **b** = **b** + **a** — is vector **a** followed by vector **b** which is equal to **b** followed by **a**.
- **a** − **b** ≠ **b** − **a** — is vector **a** followed by vector −**b** (or −**b** followed by **a**).



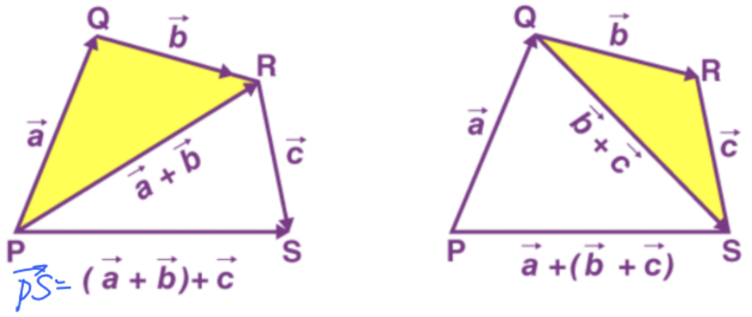
$\overrightarrow{AC}$

$\underline{b} - \underline{a} = \underline{b} + (-\underline{a})$



(2) Associative 结合律

$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$



(i) (ii)

Multiplying Vectors

Multiplying vectors by a Scalar – We can **multiply** a vector by a number. For example, we can multiply $\mathbf{a}$ by 3:

$$3\mathbf{a}=\mathbf{a}+\mathbf{a}+\mathbf{a}$$

We can **multiply more complicated vectors**

$$3(\mathbf{a}+\mathbf{b})=3\mathbf{a}+3\mathbf{b}$$

Scalar multiples – Scalar multiples are all parallel to each other: $\vec{CD}\parallel$

$AB // CD$ $\vec{AB}=$ $3\mathbf{a}+3\mathbf{b}$ is parallel to $\mathbf{a}+\mathbf{b}$

Colinear 共线

- If $\overrightarrow{AB}$ is a multiple of $\overrightarrow{BC}$, then points A, B and C are **collinear**.

Collinear points lie on the same straight line.

Combining vector addition, subtraction and multiplication

$$3a-2b=3\begin{pmatrix}5\\2\end{pmatrix}-2\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}15\\6\end{pmatrix}-\begin{pmatrix}6\\-2\end{pmatrix}=\begin{pmatrix}9\\8\end{pmatrix}$$

Example:

In triangle OAB, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

P is the point on OA with $OP:PA = 2:1$ and Q is the point on OB with $OQ:QB = 2:1$.

a Write $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

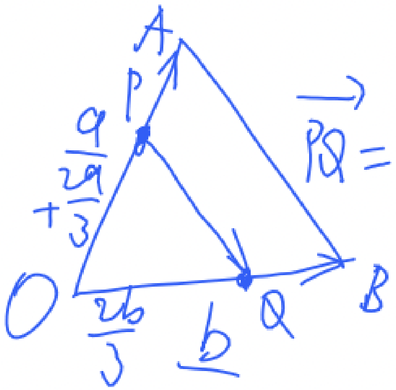
b Describe the relationship between the line segments PQ and AB.

$\vec{a}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -\underline{a} + \underline{b}$

$\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{AB}$

$PQ // AB$



$\overrightarrow{PQ} = \frac{2b}{3} - \frac{2a}{3}$

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\frac{2}{3}\underline{a} + \frac{2}{3}b$

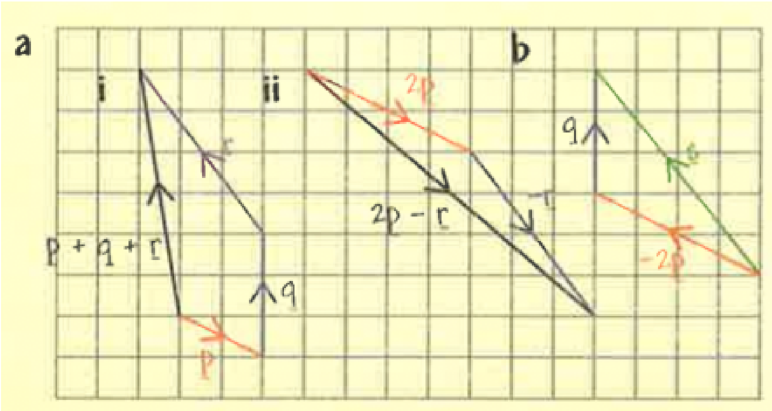
$\overrightarrow{OP} = \frac{2}{3}\underline{a} \Rightarrow \overrightarrow{PO} = -\frac{2}{3}\underline{a}$

$OQ = \frac{2}{3}\underline{b}$

Practice:

$\mathbf{p} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

a Calculate **i** $\mathbf{p} + \mathbf{q} + \mathbf{r}$ **ii** $2\mathbf{p} - \mathbf{r}$. **b** Write $\mathbf{s}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.



(b) If $p = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ $q = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$ $r = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Find:

(i) the magnitude of p

$\sqrt{(-4)^2 + 2^2} = 2\sqrt{5}$

(ii) $2r - q$ as a column vector

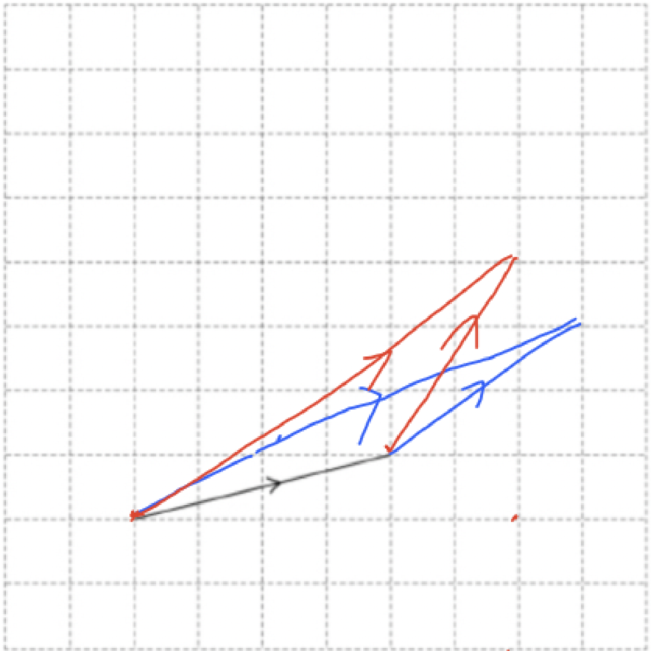
$\begin{pmatrix} -6 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$

(iii) the magnitude of $q + r$

$\begin{pmatrix} 0 \\ 7 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$ $\sqrt{9+6}$

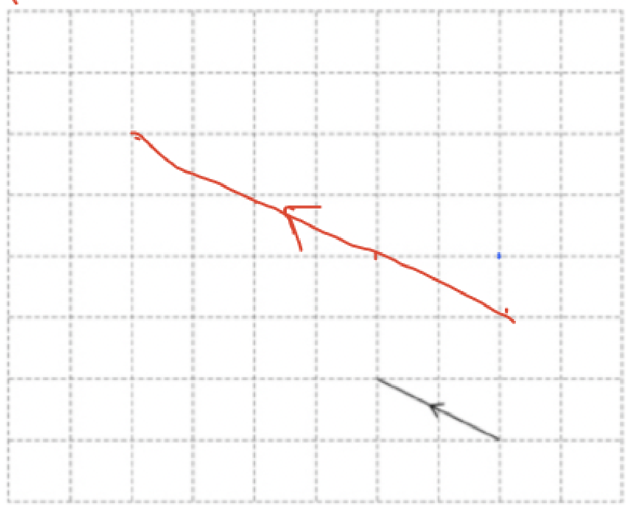
Q20N

(a) Vector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is shown on the grid.



On the grid, draw a vector that represents $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

(b) Vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is shown on this grid.



On this grid, draw a vector that is

parallel to $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

and

three times the length of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $3\sqrt{4+1} = 3\sqrt{5}$